均值模型下满足对数级收敛速度的动态系统

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摘 要 文章研究了在均值模型下，动态系统收敛至一致性状态所需的时间。在每一时间步内，节点计算其邻居的均值，并以计算结果作为自己的新值。我们考虑了当节点间的网络结构处于动态变化状态的情况。我们的分析证明了当节点的度在相邻时间点之间变化较小的情况下，即使仅满足微弱的连通性条件，动态变化的网络仍然可以保证动态系统会快速收敛到一致性状态。

关键词 对数级收敛；一致性；动态系统
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Logarithmic Convergence for Consensus of Dynamic Systems

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Abstract In this paper, the convergence time required to achieve consensus of dynamic systems was studied under the uniform averaging model. In each time step, a node's value was updated to some weighted average of its neighbors' and its old values. The case was studied when the underlying network was dynamic. Our analysis results show that dynamic networks exhibit fast convergence behavior as long as the nodes' degrees change gradually, even under very mild connectivity assumptions.

Keywords logarithmic convergence; consensus; dynamic system

1 Introduction

Many dynamic systems have been used to model the natural group behavior. Mathematically speaking, we consider the case when each individual or node in the set V maintains some real number, which can represent the opinion on some issue (e.g. preference in voting). In every time step, a node can collect the opinions of others and updates its opinion accordingly. Based on observations from many such systems[1-3], the values of all nodes converge to the consensus (a common opinion), after a small number of steps.

DeGroot[4] models the consensus of opinions using weighted averaging model. This model has been widely studied, where the interactions of the nodes in a step are modeled in a network $G_t=(V,$

E_t). An edge $\{p, q\}$ in E_t indicates that node p and node q can collect each other's number (opinion value) at time t. This weighed averaging model has applications in parallel computation[5], control theory[6-11] and ad-hoc networks[12].

Chan and Ning[13] studied what weighting strategies and what kind of networks can enable fast convergence to achieve consensus. They made a conjecture that in practical conditions, the averaging dynamic systems converged for the consensus logarithmically. In this paper, we consider a model which is essentially the same with the uniform averaging model[13], and the case when the underlying network topology is dynamic. Formally speaking, the networks G_ts change over time. In general, we only assume the general structural properties of the networks, without specifying how they evolve.

In this paper, we prove that when degrees change mildly between two consecutive networks, the dynamic system converges logarithmically.

1.1 Related Work

Researchers have studied the special case of uniform averaging model with time-invariant topology[4].

If the underlying network is time-invariant and connected, Olshevsky and Tsitsiklis[14] showed that the convergence time for the uniform averaging model was $O(n^3)$, where n denoted the number of nodes.

Relatively, researchers know little about the convergence time in the cases of dynamic networks. Cao et al.[8] showed that the convergence time was $n^{O(n)}$ assuming some special structure in the networks. Olshevsky and Tsitsiklis[14] showed that the convergence time for the uniform averaging model was $O(kn^{kn})$, with the weak connectivity

assumptions that the union of any k consecutive networks was connected. With the same weak connectivity assumption, Netic et al.[15] showed that the convergence time was $O(kn^2/\alpha)$, for a class of averaging algorithms, where α is a lower bound of the non-zeros weights involved in the algorithms.

Chan and Ning considered the cases where the connectivity was characterized via the eigenvalue gap, and showed the polynomial convergence time with mild connectivity conditions[13]. In the same paper, they proved the logarithmic convergence for a special case named static weights model.

Vicsek et al.[16] studied the interaction between particles using the weighted averaging model. The particles influence each other when they are close enough, and the convergent state is achieved when all particles travel in the same direction. Jadbabaie et al.[6] gave an explanation theoretically to such convergent behavior. Chazelle[17] studied a discrete version and showed that the convergence time was the power tower of n.

Directed networks and asynchronous updates[7] were considered. Researchers have also studied the convergence under non-linear update rules[9-11].

1.2 Our Contribution

In this paper, we focus on the uniform averaging model, and the analysis between the convergence time and the topology of the evolving networks.

Following the idea proposed by Chan and Ning[13], we measured the connectivity of a network by the eigenvalue gap of its transition matrix. By an extension of Cheeger's Inequality, it is shown that the constant eigenvalue gap implies the constant edge expansion of the network[13].

As the preparation for the main result, we firstly proved the logarithmical convergence for the special

case which corresponds to the static weight model [13]. We harnessed an idea similar to the one used in the proof[13], and presented the proof under our model description.

For the uniform averaging model, we made an analysis of the conditions on the given network sequence such that the convergence was achieved fast. With the observation that the network in practice reflects the social situation of the individuals and evolves gradually, we are motivated to introduce the property of "mildly-changing" degrees, and finally, we proved that if the sequence of networks satisfied the property of "mildly-changing" degrees, and the network at any time point was well-connected (i.e. it had constant eigenvalue gap), then the dynamic system converged to the consensus in $O(log\ n)$ time steps.

2　Preliminaries

2.1　Basic Notations

In this paper, we used $\mathbb{R}$ to denote the set of real numbers. For a vector $\boldsymbol{x}$ with entries from $\mathbb{R}$, its i-th entry is referred to using $x[i]$.

A simple undirected graph (or simply "network") is usually denoted by G, and an edge between two nodes p, q in the graph is denoted by $\{p, q\}$.

2.2　Uniform Averaging Model

Since the model we will study is essentially the same with the uniform averaging model[13] introduced, we use the same model name.

Imagine we are given n individuals and each of them holds a quantity (velocity/opinion), which can be represented by a number from $\mathbb{R}$. A configuration at some time point is an n-dimensional vector from $\mathbb{R}^n$ consisting of each individual's quantity. We denote the configuration at time t by v_t.

At each time step, the individuals are supposed to form a network G_t, in which the nodes represent the individuals and the edges reflect the association between two nodes. It is obvious that all G_ts have the same number of nodes.

Definition 1 (Dynamic System: Uniform Averaging Model)　Briefly speaking, a dynamic system indicates how to update the quantities associated with each member of a given group of individuals, according to some relation among these individuals.

In this paper, we consider the dynamic systems, in which the relations between individuals are reflected by the network G_t, and the updating process is represented by a product of a row stochastic matrix $\boldsymbol{P}_t$ and the vector consisting of the individuals' quantities, where $\boldsymbol{P}_t$, is the transition matrix of G_t, is defined as,

$$\boldsymbol{P}_t = \boldsymbol{I}_n - \boldsymbol{C}_t L_t \qquad (1)$$

In the equation, $\boldsymbol{I}_n$ is the $n \times n$ identity matrix and L_t is the Laplacian of G_t:

$$(L_t)[p,q] = \begin{cases} d_t[p] & p = q; \\ -1 & \{p,q\} \in G_t; \\ 0 & else, \end{cases} \qquad (2)$$

and $\boldsymbol{C}_t$ is a diagonal matrix, in which the i-th diagonal entry is denoted by $c_t[p]$, and satisfies

$$\frac{1}{\kappa} \leqslant c_t[p]d_t[p] \leqslant 1$$

where $\kappa \geqslant 1$ is a constant and the notation $d_t[p]$ denotes the degree of node p in G_t.

When a dynamic system is given, it actually means that the strategy to calculate $\boldsymbol{C}_t$ from a given network G_t is valid. Then the process updating configuration v_t to v_{t+1} is defined as

$$v_{t+1} = P_t v_t \qquad (3)$$

Denote a sequence of networks as

$$\mathcal{G}_s^f = \{G_t\}_{t=s}^f$$

Generally we have $s=0$ and $f=\infty$, thus we use $\mathcal{G}$ to simply denote $\mathcal{G}_0^\infty$.

In the remaining argument, we always suppose the dynamic system was already given (including the networks $\mathcal{G}$, C_ts and constant κ that defines the updating rules of v_t). Then with an initial configuration v_0 and a given sequence of network G, we want to consider when the corresponding updating process converges and try to derive an upper bound of the converging rate.

2.3 Matrix notations

As there are a lot of matrix operations involved in our argument, we introduced some notations of matrix operations in this section.

For any matrix M from $\mathbb{R}^{n \times n}$, the trace is defined as the sum of the diagonal entries, i.e.

$$tr(M) = \sum_{i=1}^n M[i,i]$$

where $M[i, i]$ denotes the i-th diagonal entry of the matrix M.

Next we introduced the τ_1 and τ_2 measures which reflect the variance between row vectors in a matrix. The τ_1-measure of M, is defined as

$$\tau_1(M) = \frac{1}{2} \max_{p,q} \left\{ \sum_k |P[p,k] - P[q,k]| \right\}$$

and the τ_2-measure of M is defined as

$$\tau_2(M) = \max_{p,q} \left(\sum_k (P[p,k] - P[q,k])^2 \right)^{\frac{1}{2}}$$

In particular, when M is of dimension $n \times 1$, i.e., M is a vector, another measure τ which doubles τ_1 is also used a lot. That is:

$$\tau(M) = \max_{p,q} |M[p] - M[q]|$$

Note that in such case (M is a vector), it holds that

$$\tau(M) = 2\tau_1(M) = \tau_2(M)$$

There are some useful facts about the τ_1 and τ_2 measures. Fact 1 states an important relationship between the τ_2-measure of the product of two matrices and product of the measures of the corresponding matrices, which is proved by Chazelle and Moreau[17, 18], fact 2 relates the τ_1-measure of a stochastic matrix with its smallest entry. Its proof is too trivial to be mentioned here.

The notation of stochastic matrix is needed for the introduction of facts. A matrix M is called stochastic if the entries sum up to 1, for every row vector in M.

Fact 1 For any stochastic matrix A and any matrix B, whose dimensions are compatible with A such that AB is well-defined, we have

$$\tau_2(AB) \leqslant \tau_1(A)\tau_2(B)$$

Based on fact 1, any stochastic matrix P has the property that $\tau_1(P) \leqslant 1$. Hence, it follows that for all ts,

$$\tau(v_{t+1}) = \tau_2(P_t v_t) \leqslant \tau(v_t)$$

Fact 2 Suppose P is a stochastic matrix so that in all its entries, there are at least some number $\alpha > 0$. Then, it holds that

$$\tau_1(P) \leqslant 1 - n\alpha$$

3 Convergence for Consensus

3.1 Simple Case: When C_t Remains the Same

In this section, at first we consider a case which is similar to the static weight model introduced by Chan and Ning[13]. Specifically, we assume C_t remains the same while the underlying network G_t changes.

Theorem 1 Provide an initial configuration v_0 and a sequence of networks $\mathcal{G}$, if all C_ts used for computing P_ts are the same, denoted by C, and the second largest (magnitude) eigenvalue of any P_t is bounded by a positive number $\lambda < 1$, then for any $\varepsilon > 0$, there is

$$t^* = \log_{\frac{1}{\lambda}} \frac{\kappa \cdot n^{\frac{3}{2}} \cdot \tau_2(v_0)}{\varepsilon}$$

such that

$$\tau_2(v_t) < \varepsilon$$

as long as $t > t^*$, where c_{max} denotes the largest element of $C^{1/2}$ and κ appears in the constraints for C_t's, i.e., for any p and t, it always holds that

$$\frac{1}{\kappa} \leqslant c_t[p]d_t[p] \leqslant 1$$

Proof Firstly, recall the facts that P_t is in the form of $I_n - CL_t$, and $c_t[p] > 0$ for all p. Being a stochastic matrix, P_t has the dominant eigenvalue $\mathbf{1}$ with right eigenvector $\mathbf{1}$ and left eigenvector

$$\pi = \left(tr(C^{-1})\right)^{-1} C^{-1}\mathbf{1}$$

Define a matrix M_t as

$$M_t = C^{-\frac{1}{2}} P_t C^{\frac{1}{2}} = I_n - C^{\frac{1}{2}} L_t C^{\frac{1}{2}}$$

Then M_t is symmetric and consequently can be diagonalized as

$$M_t = \sum_{k=0}^{n-1} \lambda_k{}^t u_k{}^t (u_k{}^t)^T$$

where $k = 0$, $u_k{}^t$'s are orthonormal eigenvectors and the eigenvalues are real.

Remark 1 Here the superscript t over the eigenvalue λ refers to the time point associated with the matrix M_t. To avoid confusing (with the "power"), we ignore this label t (or sometimes i) when it is clear from the context.

By the connectivity of G_t, stochasticity of P_t, and other properties according to the spectral graph theory, we have

$$1 = |\lambda_0| \geqslant |\lambda_1| \geqslant \cdots \geqslant |\lambda_{n-1}|$$

and

$$u_0{}^t = (\sqrt{\pi[1]}, \sqrt{\pi[2]}, \cdots, \sqrt{\pi[n]})^T$$

for every t. Because all $u_0{}^t$'s are the same, we simply write u_0 to avoid distinguishment.

Use $e(i)$ to denote the n-dimensional vector in which the i-th coordinate is 1 while the others are all zeros. Then the production of the transition matrices from time 0 to time $t-1$ is

$$P(t-1,0)[:,j] = P_{t-1}P_{t-2} \cdots P_0 e(j) =$$
$$\left(\prod_{i=t-1}^{0} C^{\frac{1}{2}} M_i C^{-\frac{1}{2}} \right) e(j) =$$
$$C^{\frac{1}{2}} \left(\prod_{i=t-1}^{0} M_i \right) C^{-\frac{1}{2}} e(j)$$

Focus on the product of M_is,

$$\prod_{i=t-1}^{0} M_i = \prod_{i=t-1}^{0} \left(\sum_{k=0}^{n-1} \lambda_k^i u_k^i (u_k^i)^T \right)$$

Each M_i can be rewritten as

$$u_0 u_0^T + \sum_{k=1}^{n-1} \left(\lambda_k u_k^i (u_k^i)^T \right)$$

denoted by $u_0 u_0^T + \hat{M}_i$. Then consider any vector x from $\mathbb{R}^n$. Next, we prove that

$$\left\| \hat{M}_i x \right\|_2 < \lambda \|x\|_2$$

Because $u_k(i)$s are orthonormal unit vectors, they form actually an orthonormal basis. Thus,

$$\left\| \hat{M}_i x \right\|_2 = \left(\sum_{k=1}^{n-1} \lambda_k^2 \left(x_k^T u_k^i \right)^2 \right)^{\frac{1}{2}} \leqslant$$
$$\lambda \left(\sum_{k=1}^{n-1} \left(x^T u_k^i \right)^2 \right)^{\frac{1}{2}} \leqslant$$
$$\lambda \left(\sum_{k=0}^{n-1} \left(x^T u_k^i \right)^2 \right)^{\frac{1}{2}} = \lambda \|x\|_2$$

Then, we rewrite $P(t-1,0)[:,j]$ as

$$P(t-1,0)[:,j] =$$

$$C^{\frac{1}{2}}\left(\prod_{i=t-1}^{0} M_i\right)C^{-\frac{1}{2}}e(j) =$$

$$C^{\frac{1}{2}}\left(\prod_{i=t-1}^{0}\left(u_0 u_0^T + \hat{M}_i\right)\right)C^{-\frac{1}{2}}e(j) =$$

$$C^{\frac{1}{2}}\left(\prod_{i=t-1}^{0} u_0 u_0^T + \prod_{i=0}^{t-1}\hat{M}_i\right)C^{-\frac{1}{2}}e(j)$$

The last equality holds because u_0 is orthogonal to any $u_k(i)$ $(k>0)$, which implies $u_0^T u_k(i)=0$. Recall that u_0 is an unit vector. Thus, we have

$$P(t-1,0)[:,j] =$$

$$C^{\frac{1}{2}}\left(\prod_{i=t-1}^{0} u_0 u_0^T + \prod_{i=t-1}^{0}\hat{M}_i\right)C^{-\frac{1}{2}}e(j) =$$

$$C^{\frac{1}{2}}u_0 u_0^T C^{-\frac{1}{2}}e(j) + C^{\frac{1}{2}}\left(\prod_{i=t-1}^{0}\hat{M}_i\right)C^{-\frac{1}{2}}e(j)$$

Note that,

$$\left\|\left(\prod_{i=t-1}^{0}\hat{M}_i\right)C^{-\frac{1}{2}}e(j)\right\|_2 \leqslant \lambda^t\left\|C^{-\frac{1}{2}}e(j)\right\|_2 \leqslant \kappa\lambda^t n^{-\frac{1}{2}}$$

and any entry of C is upper bounded by **1**. Then we get

$$\tau_1\big(P(t-1,0)[:,j]\big) =$$

$$\frac{1}{2}\max_{p,q}\big|P(t-1,0)[p,j] - P(t-1,0)[q,j]\big| \leqslant$$

$$\kappa\lambda^t n^{\frac{1}{2}}$$

which implies

$$\tau_1\big(P(t-1,0)\big) \leqslant \sum_{j=1}^{n}\tau_1\big(P(t-1,0)[:,j]\big) \leqslant \kappa\lambda^t n^{\frac{3}{2}}$$

Thus we finally get

$$\tau_2(v_t) \leqslant \tau_1\big(P(t-1,0)\big)\tau_2(v_0) \leqslant \kappa\lambda^t n^{\frac{3}{2}}\tau_2(v_0)$$

Hence, $\tau_2(v_t)$ goes below ε, for $t>t^*$, which is

$$t^* = \log_{\frac{1}{\lambda}}\frac{\kappa\cdot n^{\frac{3}{2}}\cdot\tau_2(v_0)}{\varepsilon}.$$

3.2 General Case: When C_t Changes

Chan and Ning[13] made a conjecture that in practical conditions, the averaging dynamic systems converged for the consensus logarithmically. In this section, we introduced a condition which was mild and appeared normally in practice, and then proved that under such a condition, the conjecture made by Chan and Ning[13] is true.

Condition 1 (mildly changing degree) Provide a sequence of networks $\mathcal{G}$, if $\mu>0$, then for any time t and node p, it holds that

$$\max_{i,p}\frac{d_{i+1}[p]}{d_i[p]} \leqslant \mu$$

then we say that the given network sequence $\mathcal{G}$ has "mildly-changing" degrees.

Theorem 2 Provide an initial configuration v_0 and a sequence of networks $\mathcal{G}$ which has "mildly-changing" degrees, and the second largest (magnitude) eigenvalue of $\boldsymbol{P}_t$ is bounded by a positive number $\lambda<1$ for every $t>0$, then the dynamic system converges for the consensus at a rate of order $O(\log n)$.

Proof At first, recall that

$$P(t-1,0)v_0 =$$

$$P_{t-1}P_{t-2}\cdots P_0 v_0 = -\left(\prod_{i=t-1}^{0} C_i^{\frac{1}{2}}M_i C_i^{-\frac{1}{2}}\right)v_0$$

and

$$M_i = u_0^i(u_0^i)^T + \sum_{k=1}^{n-1}\lambda_k u_k^t(u_k^t)^T$$

To simplify the presentation, use the notation

$$F_i = C_i^{\frac{1}{2}}u_0^i(u_0^i)^T C_i^{-\frac{1}{2}}$$

and

$$D_i = C_i^{\frac{1}{2}}\left(\sum_{k=1}^{n-1}\lambda_k u_k^t(u_k^t)^T\right)C_i^{-\frac{1}{2}}$$

Note that for any i,

$$C_i^{\frac{1}{2}} u_0^i = \frac{\mathbf{1}}{\left(tr C_t^{-1}\right)^{\frac{1}{2}}}$$

Thus, $\mathbf{1}$ is orthogonal to $C_i^{-\frac{1}{2}} u_k^i$ for any i and $k>0$, which implies $D_i F_j = 0$ for any i and j. Then we have

$$P(t-1,0)v_0 =$$

$$\left(\prod_{i=t-1}^{0} (F_i + D_i)\right) v_0 =$$

$$\left(\prod_{i=t-1}^{2} (F_i + D_i)\right)(F_1 F_0 + F_1 D_0 + D_1 D_0) v_0 =$$

$$\cdots$$

$$\left(\sum_{i=t-1}^{0}\left(\prod_{p=t-1}^{i} F_p \prod_{q=t}^{0} D_q\right)\right) v_0 + \left(\prod_{i=t-1}^{0} D_i\right) v_0$$

Note that the first part

$$\left(\sum_{i=t-1}^{0}\left(\prod_{p=t-1}^{i} F_p \prod_{q=t}^{0} D_q\right)\right) v_0$$

is a vector along the direction 1, thus it decides the consensus which is finally achieved, if the remaining part $\left(\prod_{i=t-1}^{0} D_i\right) v_0$ converges to 0. Note that

$$\left(\prod_{i=t-1}^{0} D_i\right) v_0 =$$

$$C_{t-1}^{1/2} \hat{M}_{t-1}\left(\prod_{i=t-2}^{0} C_{i+1}^{-1/2} C_i^{1/2} \hat{M}_t\right) C_0^{-1/2} v_0$$

where $\hat{M}_i = \sum_{k>0} \lambda_k^i u_k^i (u_k^i)^T$. The (p,q)-th entry of $C_{i+1}^{-1/2} C_i^{1/2}$ satisfies

$$C_{i+1}^{-1/2} C_i^{1/2}[p,q] = \begin{cases} 0 & p \neq q \\ \dfrac{c_i[p]}{c_{i+1}[p]} & p = q \end{cases}$$

and recall that

$$\frac{1}{\kappa} \leqslant c_t[p] d_t[p] \leqslant 1$$

Thus we have

$$\frac{c_i[p]}{c_{i+1}[p]} \leqslant \frac{\kappa d_{i+1}[p]}{d_i[p]}$$

Recall the "mildly-changing" property of the degrees, i.e. there exists $\mu>0$, then

$$\max_{i,p} \frac{d_{i+1}[p]}{d_i[p]} \leqslant \mu$$

Hence we finally get

$$\left\| C_{i+1}^{-1/2} C_i^{1/2} \hat{M}_i x \right\|_2 \leqslant \kappa\mu \left\| \hat{M}_i x \right\|_2 \leqslant \kappa\mu\lambda \|x\|_2$$

which implies that

$$\left\| \left(\prod_{i=t-1}^{0} D_i\right) v_0 \right\|_2 < \varepsilon$$

for any $\varepsilon>0$, and $t>t^*$, where $t^* = O(log\ n)$. The detailed analysis of t^* is ignored since it is similar to the proof of theorem 1.

4　Conclusions

In this paper, we confirmed the conjecture raised by Chan and Ning[13] that concluded the logarithmic convergence for consensus of dynamic systems, under the "mildly-changing" degree condition. As in practice, the degree of a node reflects the personal social situation, which usually evolves gradually, the property of "mild-changing" degree is easily satisfied. Hence, our results help people to understand the fast convergence of group behavior in natural world, like bird flocking.

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